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# ANALYTICAL SOLUTION OF A NONHOMOGENEOUS KINETIC EQUATION WITH A VARIABLE FREQUENCY OF COLLISIONS

**A. V. Latyshev, V. N. Popov, and A. A. Yushkanov**

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*The problem of calculation of a correction to the coefficient of isothermal slip that is due to the wall curvature is solved. For this purpose we constructed the exact solution of a nonhomogeneous model kinetic Boltzmann equation with a collision operator in the form of the BGK model with the frequency of collisions proportional to the modulus of the intrinsic velocity of gas molecules. Comparison with the results obtained earlier is given.*

**Introduction.** Application of model equations to solution of the boundary-value problems of the kinetic theory of gases is rather traditional now. A large number of works are devoted to such studies (see, e.g., [1–3] and the references therein). In most of these works, use is made of model equations with a constant frequency of collisions. However, the assumption of independence of the frequency of collisions of gas molecules from their velocity is a strong simplification. The assumption of constancy of the mean free path of molecules seems to be more realistic. It is equivalent to the fact that the frequency of collisions of molecules is proportional to the absolute value (modulus) of their velocity.

The present work is aimed at constructing the solution of a nonhomogeneous kinetic Boltzmann equation with a collision operator in the form of the Bhatnagar–Gross–Krook (BGK) model with the frequency proportional to the modulus of the velocity of gas molecules in the problem of isothermal slip of a rarefied gas along a solid spherical surface. Additionally, we calculate the correction $K_{\text{sl}}^{(1)}$ to the coefficient of isothermal slip that is due to the wall curvature.

**Basic Equations and Their Solution.** Calculation of the sought correction is reduced to solution of the equation

$$C_r \frac{\partial Y^{(2)}}{\partial r} + CY^{(2)}(r, \theta, C_i) = \frac{\sqrt{\pi}}{2} C \int \rho(C') K_{\text{f}}(C_i, C_i') Y^{(2)}(r, \theta, C_i') d^3 C_i' -$$

$$- (C_\theta^2 + C_\varphi^2) \frac{\partial Y^{(1)}}{\partial C_r} - (C_\varphi^2 \operatorname{ctan} \theta - C_r C_\theta) \frac{\partial Y^{(1)}}{\partial C_\theta} + (C_\varphi C_\theta \operatorname{ctan} \theta + C_r C_\varphi) \frac{\partial Y^{(1)}}{\partial C_\varphi} - C_\theta \frac{\partial Y^{(1)}}{\partial \theta}, \quad (1)$$

$$K_{\text{f}}(C, C') = 1 + \frac{3}{2} C_i C_i' + \frac{1}{2}(C^2 - 2)(C'^2 - 2), \quad \rho(C') = \pi^{-3/2} C \exp(-C^2)$$

with the boundary conditions

$$Y^{(2)}(R, \theta, C_i) = -2C_\theta U_\theta^{(2)}\big|_S, \quad C_r > 0; \quad Y^{(2)}(\infty, \theta, C_i) = 0.$$

We take into account that in the case of isothermal slip of the rarefied gas along the spherical surface

Pomor'e State University, Arkhangel'sk, Russia; email: popov.vasily@pomorsu.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 75, No. 3, pp. 104–106, May–June, 2001. Original article submitted November 26, 2001; revision submitted December 3, 2001.

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$$Y^{(j)}(r, \theta, C_i) = C_\theta Z^{(j)}(r, \theta, C_r), \quad j = 1, 2 . \tag{2}$$

We substitute (2) into (1) and pass to the spherical coordinate system in the velocity space

$$C_r = C \cos \xi ; \quad C_\theta = C \sin \xi \cos \zeta ; \quad C_\varphi = C \sin \xi \cos \zeta .$$

We introduce the notation $\tau = \cos \xi$. Then, multiplying the obtained equation by $\cos \zeta$ and integrating with respect to $\zeta$ from 0 to $2\pi$, we come to the equation

$$\tau \frac{\partial Z^{(2)}}{\partial r} + Z^{(2)}(r, \theta, \tau) = \frac{3}{4} \int_{-1}^{1} (1 - \tau'^2) Z^{(2)}(r, \theta, \tau') d\tau' - (1 - \tau^2) \frac{\partial Z^{(1)}}{\partial \tau} + \tau Z^{(1)}(r, \theta, \tau) \tag{3}$$

with the boundary conditions

$$Z^{(2)}(R, \theta, \tau) = -2U_\theta^{(2)}\big|_S , \quad 0 < \tau < 1 , \quad Z^{(2)}(\infty, \theta, \tau) = 0 . \tag{4}$$

We present the general solution of Eq. (3) without derivation:

$$Z^{(2)}(x, \theta, \tau) = (1 + x)\, \varphi(x, \theta, \tau) - \frac{3}{4} \int_0^1 \exp(-x/\eta)\, \Phi_{\mathrm{f}}(\eta, \theta, \tau)\, d\eta + \frac{x}{\tau(1 - \tau^2)} \frac{a(\tau, \theta)}{\tau} \exp(-x/\tau)\, \Theta(\tau) , \tag{5}$$

$$\varphi(x, \theta, \tau) = A + B(x - \tau) + \int_{-1}^{1} \exp(-x/\eta)\, F_{\mathrm{f}}(\eta, \tau)\, n(\eta, \theta)\, d\eta ,$$

$$F_{\mathrm{f}}(\eta, \tau) = \frac{3}{4} \eta P \frac{1}{\eta - \tau} + \frac{\lambda_{\mathrm{f}}(\eta)}{1 - \eta^2} \delta(\eta - \tau) ,$$

$$\Phi_{\mathrm{f}}(\eta, \theta, \tau) = [\eta a'(\eta, \theta) + (1 - \tau^2)\, a(\eta, \theta)] P \frac{1}{\eta - \tau} + g(\eta, \theta)\, \delta(\eta - \tau) ,$$

$$g(\eta, \theta) = \frac{4}{3} \frac{1}{1 - \eta^2} [(\eta a(\eta, \theta))' (\lambda_{\mathrm{f}}(\eta) - 1)/\eta - \eta a(\eta, \theta)\, \lambda_{\mathrm{f}}(\eta)] ,$$

$$a_{\mathrm{f}}(\eta, \theta) = \frac{2\eta (1 - \eta^2)\, X(-\eta)}{5\, |\lambda_{\mathrm{f}}^{+}(\eta)|^2} k , \quad X_{\mathrm{f}}(z) = \frac{1}{z} \exp V_{\mathrm{f}}(z) ,$$

$$V_{\mathrm{f}}(z) = \frac{1}{\pi} \int_0^1 \frac{\theta_{\mathrm{f}}(\eta) - \pi}{\eta - z} d\eta , \quad \theta_{\mathrm{f}}(\eta) - \pi = -\pi/2 - \arctan \frac{4\lambda_{\mathrm{f}}(\eta)}{3\pi\eta(1 - \eta^2)} ,$$

$$\lambda_{\mathrm{f}}(\eta) = 1 + \frac{3}{4} \eta \int_{-1}^{1} \frac{1 - \tau^2}{\tau - \eta} d\tau , \quad x = r - R , \quad k = \frac{\partial U_\theta}{\partial r}\bigg|_S .$$

The parameters $A$, $B$, and $n(\eta, \theta)$ which enter into (5) are determined from boundary conditions (4) with the use of the theory of boundary-value problems [4–6]. Here we find

$$U_\theta^{(2)}\big|_S = \frac{3}{8}\left[\int_0^1 \eta a\,(\eta, \theta)\, d\eta + 5\int_0^1 \frac{(\eta a\,(\eta, \theta))'}{X\,(-\eta)}\, d\eta\right]. \tag{6}$$

The integrals which enter into (6) are evaluated analytically according to the technique of [4–6]. Substituting the obtained values of the integrals into (6), we find

$$U_\theta^{(2)}\big|_S = -\,0.2857143\, k\,. \tag{7}$$

Passing in (7) to dimensional quantities [4, 5] and allowing for the fact (see [5]) that $K_{\mathrm{sl}}^{(0)} = 1.0922$ and $R^{-1} = 15\mathrm{Kn}/8$, we obtain

$$U_\theta^{(2)}\big|_S = -\,0.9196812\ \mathrm{Kn}\ \lambda k\,.$$

Thus, $K_{\mathrm{sl}}^{(1)} = 0.9196812$.

**Conclusions.** We have developed a method which leads to the exact solutions of boundary-value problems for nonhomogeneous model kinetic equations with the frequency proportional to the absolute value of the velocity of gas molecules. In addition, we have calculated the correction $K_{\mathrm{sl}}^{(1)}$ which allows for the dependence of $K_{\mathrm{sl}}$ on the radius of curvature of the phase interface. It follows from the results of the work that account for the dependence of the frequency of collisions on the modulus of the absolute velocity of gas molecules leads to a decrease of 17.18% in $K_{\mathrm{sl}}^{(1)}$ compared to the similar result obtained with the use of the BGK model of the kinetic Boltzmann equation with a constant frequency of collisions ($K_{\mathrm{sl}}^{(1)} = 1.110418$). Earlier, in [4, 5], it was noted that such account for the dependence of the frequency of collisions on the modulus of the absolute velocity of gas molecules leads to a decrease of 4.74% in $K_{\mathrm{sl}}^{(0)}$.

## NOTATION

$R$, radius of the sphere; $S$, surface; Kn, Knudsen number, $U_\theta$, component of the gas velocity that is tangential to the surface; $U_\theta\big|_S$, gas velocity on the surface; $K_{\mathrm{sl}}$, coefficient of isothermal slip; $K_{\mathrm{sl}}^{(0)}$, coefficient of isothermal slip of the rarefied gas along the solid plane surface; $K_{\mathrm{sl}}^{(1)}$, coefficient which allows for the dependence of the coefficient of isothermal slip on the radius of curvature of the surface; $k$, gradient of mass velocity on the surface; $C_r$, $C_\theta$, and $C_\varphi$, components of the dimensionless intrinsic velocity of molecules written in the spherical coordinate system; $r$, modulus of the dimensionless radius vector; $\theta$ and $\varphi$, angular coordinates of the spherical coordinate system; $C$, modulus of the dimensionless intrinsic velocity of molecules; $x$, distance reckoned along the normal from the sphere surface; $Z^{(1)}(r, \theta, C_i)$, solution of the Kramers problem (problem of isothermal slip of a rarefied gas along a solid plane surface) constructed in [4, 5] with the use of the considered model of the kinetic Boltzmann equation; $\xi$ and $\zeta$, angular coordinates of the spherical coordinate system in the velocity space; $\eta$, spectral parameter of decomposition; $F_{\mathrm{f}}(\eta, \tau)$, eigenvectors of the continuous spectrum from the Kramers problem constructed in [4, 5] with the use of the considered model of the kinetic Boltzmann equation; $\Phi_{\mathrm{f}}(\eta, \tau)$, eigenvectors of the continuous spectrum from the problem of isothermal slip of the rarefied gas along the solid spherical surface constructed with the use of the considered model of the kinetic Boltzmann equation; $\lambda_{\mathrm{f}}(z)$, dispersion function obtained in [4, 5] with the use of the considered model of the kinetic Boltzmann equation; $Px^{-1}$, distribution in terms of the principal value in evaluating the integral of $x^{-1}$; $\delta(x)$, Dirac delta function; $a_{\mathrm{f}}(\eta, \theta)$, coefficients in the expansion in eigenvectors of the continuous spectrum of the solution of the Kramers problem, which are constructed in [4, 5] with the use of the considered model of the kinetic Boltzmann equation; $X_{\mathrm{f}}(z)$, canonical function from the Kramers problem, which is constructed in [4, 5] with the use of the considered model of the kinetic Boltzmann equation; $Y$, function allowing for the deviation of the distribution function in the Knudsen layer from the distribution function in the gas volume; $\theta_{\mathrm{f}}(\eta)$, single-valued regular branch of the argument of the function $\lambda_{\mathrm{f}}^{+}(\tau)$ fixed at zero by the condition $\theta(0) = 0$; $\Theta(\tau)$, Heaviside step function. Sub- and superscripts: r, $\theta$, and $\varphi$, projections to the axes of the spherical coordinate system; sl, slip; f, frequency; (0), (1), and (2), ordinal numbers of the corresponding coefficients in the expansion of quantities into a series in powers of the inverse radius; $'$, integration variable; +, boundary value of the function of the complex variable on the upper edge of the cut.

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